Functions of several variables

(M.Sc. (MATHEMATICS), Paper - VI)

(Real Analysis - II)

Lecture - 01

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Paper-V

Functions of Two Variables

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$:→

A linear transformations

$$T : V = \mathbb{R}^n(\mathbb{R}) \longrightarrow W = \mathbb{R}^m(\mathbb{R})$$

from a vector space V into another vector space W. (Where V and W are finite-dimensional vector space)

→ If $n = m = 1$ i.e; $f : \mathbb{R} \to \mathbb{R}$ is called a real-valued function of a real variable.

→ When $n = 1$, and $m > 1$ i.e; $f : \mathbb{R} \to \mathbb{R}^m$ is a vector valued function of a real variable.

→ When $n > 1$ and $m \geq 1$.

(i) If $m = 1$, the function is called a real-valued function of a vector variable or a scalar field. (i.e; $f : \mathbb{R}^n \to \mathbb{R}$)

(ii) If $m > 1$ it is called a vector-valued function of a vector variable or simply a vector field. (i.e; $f : \mathbb{R}^n \to \mathbb{R}^m$).

Definition: $\rightarrow$ If $x, y \in \mathbb{R}^n$, then inner Product is defined as

If $x = (x_1, ----, x_n)$ & $y = (y_1, --- y_n) \in \mathbb{R}^n$

$$\langle x, y\rangle = x \cdot y = \sum_{k=1}^{n} x_k y_k = x_1 y_1 + ------ + x_n y_n .$$

And corresponding norm is denoted by $\|x\|$ and defined as

$$\|x\| = \sqrt{\langle x, x\rangle} = (x \cdot x)^{\frac{1}{2}} = \sqrt{x_1^2 + --- + x_n^2} .$$

Open ball and Open set: $\rightarrow$

Let $a = (a_1, ----, a_n)$ be a given point in $\mathbb{R}^n$ and $r > 0$ be a given positive real number.

The set of all points $x = (x_1, -----, x_n) \in \mathbb{R}^n$

s.t. $\|x - a\| < r$

is called an open n – ball of radius r and center a.

We denote this set by $B(a)$ or $B(a, r)$.

Example: $\rightarrow$ (i) In $\mathbb{R}$ this is simply an open interval $(a-r, a+r)$.

i.e; $B(a; r) = (a-r, a+r)$.

(ii) In $\mathbb{R}^2$ it is a circular disk

i.e; $B(a, r) = \{ x = (x_1, y_1) \in \mathbb{R}^2 : \|x - a\| < r \}$

Where $a = (a_1, a_2) \in \mathbb{R}^2$.

Then $\|x - a\| = \|(x_1 - a_1, x_2 - a_2)\|$

$$= \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$$

i.e; $B(a, r) = \{ x \in \mathbb{R}^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2 \}$

(iii) In $\mathbb{R}^3$ it is a spherical solid with ~~center~~ center $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ & radius $r > 0$.

Definition (interior Point): → Let S be a subset of $\mathbb{R}^n$, and assume that $a \in S$. Then a is called an interior point of S if there is an open n-ball with center a, such that $B(a, r) \subseteq S$.

Remark: → ① The set of all interior points of S is called interior of S, and denoted by int S.

② An open set containing a point 'a' is sometimes called neighbourhood of 'a'

Definition (OPEN SET) :→ A set S in $\mathbb{R}^n$ is called open if all its points are interior points.

OR

$S \subseteq \mathbb{R}^n$ is open if and only if $S = \text{int} S$.

Definition (Exterior and Boundary) :→ A point $a \in \mathbb{R}^n$ is said to be exterior point of a set S in $\mathbb{R}^n$ if $\exists$ an open ~~ball~~ n-ball $B(a)$ containing no points of S.

The set of all points in $\mathbb{R}^n$ exterior to S is called the exterior of S and denoted by ext S.

A point which is neither exterior to S nor an interior point of S is called a boundary point of S.

The set of all ~~points of~~ boundary points of S is called the boundary of S and is denoted by ∂S.